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Self-similar evolution of a twin boundary in anti-plane shear

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Abstract. This paper studies the transient motion of a twin boundary in two dimensions. The twinning deformation is described as an anti-plane shear deformation with discontinuous strains. The material is assumed to be compressible and hyperelastic with a stored energy function consisting of multiple potential wells. The quasi-steady-state evolution of a twinning step is studied. The model includes an anisotropic kinetic relation that governs the twin boundary motion in two dimensions under applied stress. A self-similar solution for the motion of the twinning step is found with a specific initial shape. General solutions to the linearized evolution equation are established in the form of an infinite series for arbitrary initial shapes. Stability of the self-similar solution is discussed.

Key words: nonlinear elasticity, anti-plane shear, twin-boundary motion, quasi-steady-state propagation, integro-partial-differential equation.

1. Introduction

Under applied loading, crystals often undergo deformations that develop microstructure with lower energy level. Deformation mechanisms such as twinning and phase transformation allow the crystal to accommodate large deformations. Microstructure involving layers of twins is a prominent feature of stress-induced phase transformations. They are often observed during the formation of austenite-martensite interfaces. The evolution of these twin boundaries is responsible for the shape-memory effects of certain alloys and plays a key role in the plastic deformations of metals under high velocity loadings [1].

In the state produced by twinning, a part of the material suffers a large shear relative to the rest. The two portions are separated by a sharp interface called the twin boundary. Special shear strains cause both sides to exhibit almost identical lattice structures with difference only in their orientation. The twin boundaries are in general coherent; *i.e.*, the deformation is continuous but the strains have a finite jump across the twin boundaries.

A continuum-mechanical theory capable of modeling deformation twinning and various phase transformations is derived through the works of Ericksen [2, 3], Knowles and Sternberg [4, 5, 6], James [7, 8, 9], Gurtin [10] and Abeyaratne and Knowles [11, 12, 13]. The approach is to model the deformations by elastic fields with discontinuous strains across the boundaries. The stored energy function consists of multiple wells, each corresponding to a phase or variant. The strains on each side of the phase boundaries must stay relative close to the well in order to avoid the unstable regions that separate the wells in the strain space.

For twinning deformations, additional considerations must be taken to account for the special symmetry associated with twinning. For this purpose, Rosakis and Tsai [14] proposed a nonlinear elastic constitutive law for body-centered cubic crystals. The resulting

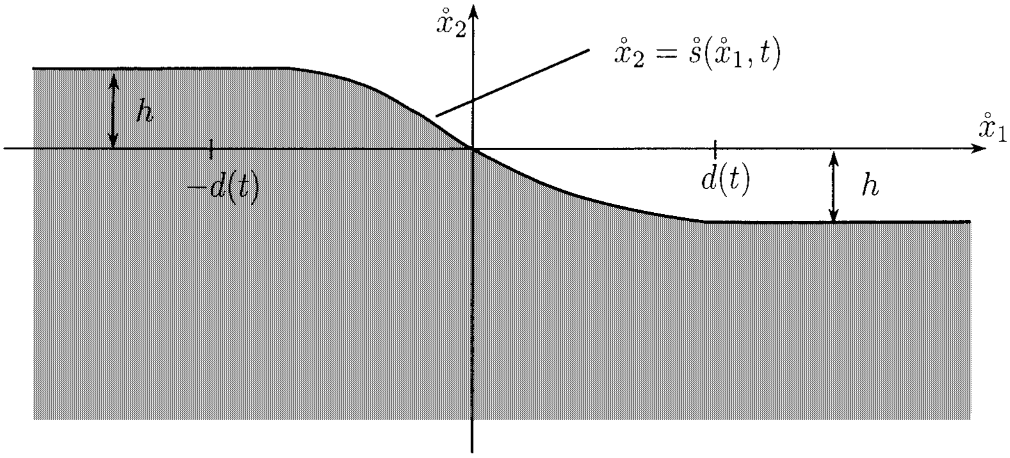

Figure 1. Typical shape of a twinning step. The shaded area indicates material in variant $\mathcal{S}_1$, the unshaded area $\mathcal{S}_0$. The curved portion of the twin boundary is in the interval $\mathring{x}_1 \in (-d(t), d(t))$. The moving frame $\mathring{x}_1$–$\mathring{x}_2$ has the velocity $\mathring{V}\mathbf{e}_1$.

stored energy function possesses multiple wells, each corresponding to a twin variant, and embodies regions of unstable shears associated with failure of ellipticity. The structure of the mechanical behavior for anti-plane shear is deduced from considerations of lattice symmetry by Tsai [15] and is consistent with the twinning modes in BCC lattice. Based on this model, the displacement fields associated with a twin needle in equilibrium as well as its steady-state propagation are obtained [15, 16]. Similar to the dynamic phase boundary propagation in one-dimensional bar [17, 18], the motion of the twin boundaries is not determined. The lack of uniqueness can be overcome by imposing a kinetic relation governing the normal velocity of the moving boundary. A special anisotropic kinetic relation is proposed by Rosakis and Tsai [16] to account for the preferred orientation of the twin boundary. They conclude that under the special kinetic assumption the subsonic steady-state propagation of a twin needle can not occur, while one can construct a displacement field for the supersonic growth provided the critical applied stress is exceeded. Therefore, the subsonic growth of twin needles is not a steady state and must be transient in nature.

In this paper, we consider the transient motion of a twinning step under applied loading in the setting of anti-plane shear. We formulate the quasi-steady-state motion of a twinning step which is aligned with the preferred orientation of the twinning structure except over a bounded and curved region (see Figure 1). The twinning step is moving toward its axial direction with small variation from the average propagating velocity. With respect to a moving frame at the average velocity, the motion of the twin boundary with respect to this frame is assumed to be quasi-static. Under this approximation, a closed form solution for the displacement field is obtained. We study the implication of a special anisotropic kinetic relation which relates the normal velocity to the driving force as well as the orientation on the twin boundary. The evolution of the twin boundary is found to be governed by an integro-differential equation. We seek a self-similar solution which approaches but never achieves a steady state as time increases. This self-similar solution consists of a special initial shape. We also analyze the linearization of the evolution equation in the case that the twin boundary assumes an arbitrary initial shape. The stability of the self-similar solution and the limitation of the linearization are discussed.

The outline of the paper is as follows. The basic formulation of the anti-plane shear model of twinning is described in Section 2. The fundamental displacement field is presented in Section 3. Displacement field associated with the quasi-steady-state motion of a twinning step is established in Section 4. In Section 5, the kinetic relation is used to derive the governing equation for the twinning step evolution. A self-similar solution is found for a specific initial shape of the twin boundary. Linearization of the evolution equation is performed in Section 6 to study the general motion of twinning steps.

2. Anti-plane shear formulation of twinning deformations

Consider a body which occupies a cylindrical region $\mathcal{R}$ in its reference configuration. Define an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3$ along the axis of $\mathcal{R}$. The cross section is denoted by the two-dimensional region Π on the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. The vector $\mathbf{y}$ is the position of a material point that occupies the position $\mathbf{x}$ in the reference configuration. An anti-plane shear motion of the body is characterized by the deformation:

$$\mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) = \mathbf{x} + u(x_1, x_2, t)\mathbf{e}_3 \qquad \forall \mathbf{x} \in \mathcal{R},\ t \in \mathcal{T}, \tag{2.1}$$

where u is the out-of-plane displacement field defined on the cross-section Π on the (x_1, x_2) plane. Here we assume that u is smooth in t and at least twice continuously differentiable in Π except possibly on a collection of piecewise smooth surfaces Σ_t. We define the two-dimensional shear strain vector $\boldsymbol{\gamma}$ as follows:

$$\boldsymbol{\gamma} = \gamma_\alpha \mathbf{e}_\alpha = u_{,\alpha}\mathbf{e}_\alpha. \tag{2.2}$$

The shear strain is allowed to have jump discontinuities across Σ_t, while keeping the displacement continuous. Notice that the surfaces Σ_t might move through the cross-section Π during a dynamic process; their motion is determined by the scalar normal velocity V_n. The continuity of displacement across Σ_t requires the following jump conditions be valid on Σ_t:

$$[\![u_{,\alpha}]\!] l_\alpha = 0, \tag{2.3}$$

$$[\![\dot{u}]\!] + [\![u_{,\alpha}]\!] n_\alpha V_n = 0, \tag{2.4}$$

where $\mathbf{n} = n_\alpha \mathbf{e}_\alpha$ is the unit normal to Σ_t and $\mathbf{l} = l_\alpha \mathbf{e}_\alpha$ is the unit tangent to Σ_t on the (x_1, x_2) plane.

For a class of elastic materials, the full three-dimensional equations of linear momentum balance are reduced to a single equation involving the out-of-plane displacement u and the shear stress components $\sigma_{3\alpha}$ of the Piola-Kirchhoff stress tensor:

$$\sigma_{3\alpha,\alpha} = \rho\ddot{u}, \qquad \text{on } \Pi - \Sigma_t, \tag{2.5}$$

where the mass density ρ is assumed to be constant. The other two in-plane equations are automatically satisfied when certain restrictions on the material constitution are imposed. A discussion of dynamic anti-plane shear for this special class of materials in a three-dimensional setting can be found in [19]. On Σ_t, the balance of linear momentum reduces to the traction jump condition:

$$[\![\sigma_{3\alpha}]\!] n_\alpha + \rho[\![\dot{u}]\!] V_n = 0 \qquad \text{on } \Sigma_t. \tag{2.6}$$

The material under consideration is assumed to be compressible and hyperelastic. For anti-plane shear deformations, the store energy density can be reduced to a function of the shear strains $\boldsymbol{\gamma}$. Denoting the stored energy function by $w(\gamma_1, \gamma_2)$, we may give the shear stresses $\sigma_{3\alpha}$ by the following constitutive relation

$$\sigma_{3\alpha} = \frac{\partial w}{\partial \gamma_\alpha}(\gamma_1, \gamma_2). \tag{2.7}$$

Twinning in crystals involves a planar twin boundary across which the strain suffers discontinuities. On one side, the material is deformed by a simple shear along a direction parallel to the twin boundary. The resulting lattice structures on both sides of the twin boundary are identical but differ in orientation by a reflection or a 180° rotation. The direction and the amount of the shear can be determined once the geometry of the original lattice structure is known. The boundary is coherent in the sense that the displacements remain continuous across the twin boundary, while the sharp interface indicates a jump in the strains. A continuum model of twinning in the content of finite elasticity has been developed by Ericksen [20], James [9] and Pitteri [21]. A constitutive model for BCC crystals incorporating the lattice geometry is constructed by Rosakis and Tsai [14]. By choosing the twinning shear direction [111] to be the out-of-plane direction $\mathbf{e}_3$, we can describe the finite shear associated with twinning as an anti-plane shear deformation. There are three possible directions of the (112) type for the normal of the corresponding twin boundaries. If one of them is taken to be along $\mathbf{e}_2$, the twinning shear vectors corresponding to these three twinning modes are $\boldsymbol{\gamma} = \boldsymbol{\xi}^i$, $i = 1, 2$ or 3. Since the amount of shear is $1/\sqrt{2}$, letting $\boldsymbol{\xi}^0 = \mathbf{0}$, the components of these vectors are:

$$\begin{aligned} &(\xi_1^0, \xi_2^0) = (0, 0), && (\xi_1^1, \xi_2^1) = \xi(0, 1), \\ &(\xi_1^2, \xi_2^2) = \xi\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), && (\xi_1^3, \xi_2^3) = \xi\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \qquad \xi = \frac{1}{\sqrt{2}}. \end{aligned} \tag{2.8}$$

As shown by Rosakis and Tsai [14], the stored energy function w must have global minima at $\boldsymbol{\gamma} = \boldsymbol{\xi}^0, \boldsymbol{\xi}^1, \boldsymbol{\xi}^2$ and $\boldsymbol{\xi}^3$. The material is stress-free when the shear strains are at these values. Around each minimum, there is a disjoint region within which the energy function is convex. Outside these regions, the energy is not convex and hence is considered unstable. For simplicity, assume these regions are given by

$$\mathcal{S}_i = \{\boldsymbol{\gamma} \mid \ |\boldsymbol{\gamma} - \boldsymbol{\xi}^i| < \delta\}. \tag{2.9}$$

These regions on the shear strain plane (γ_1, γ_2) are referred to as variants; each is a disk of radius $\delta < \xi/2$ centered at the corresponding energy minimum. For reasons of stability, the strains can only take values in these variants. For a deformation involving shear strains in two different variants, a twin boundary forms to separate materials associated with strains in different variants. The strains are necessarily discontinuous across the boundary and must satisfy the jump conditions (2.3), (2.4) and (2.6).

A specific stored energy function that is consistent with BCC symmetry is given by

$$w(\boldsymbol{\gamma}) = \frac{\mu}{2}|\boldsymbol{\gamma} - \boldsymbol{\xi}^i|^2, \quad \boldsymbol{\gamma} \in \mathcal{S}_i, \qquad i = 0, 1, 2 \text{ or } 3, \tag{2.10}$$

where μ is the shear modulus. This energy density function is isotropic in each variant. Since the lattice structures are identical at the center of each variant, the shear moduli must be the same. In the subsequent analysis, we adopt this constitutive model and assume that the stored energy density be given by (2.10). Also, we restrict the consideration on twinning deformations formed by the variant pair $\mathcal{S}_0$ and $\mathcal{S}_i$. With the stored energy density (2.10), the equation of motion (2.5), in terms of the out-of-plane displacement u, reduces to the wave equation:

$$\nabla^2 u = \frac{1}{c^2}\ddot{u} \quad \text{on } \Pi - \Sigma_t, \qquad c = \sqrt{\frac{\mu}{\rho}}. \tag{2.11}$$

On the twin boundary Σ_t, the following jump conditions must be valid:

$$\begin{cases} [\![u_{,\alpha}]\!] l_\alpha = 0, \\ [\![\dot{u}]\!] + [\![u_{,\alpha}]\!] n_\alpha V_n = 0, \\ [\![u_{,\alpha}]\!] n_\alpha + \frac{V_n}{c^2}[\![\dot{u}]\!] = [\![\xi^i_\alpha]\!] n_\alpha. \end{cases} \qquad \text{on } \Sigma_t. \tag{2.12}$$

It is now well known [22, 23] that during a deformation process, the total mechanical energy associated with an elastic material may dissipate due to the presence of moving surfaces across which the strains suffer jump discontinuities even though the material is elastic. Under the current setting, the energy dissipation rate due to a moving twin boundary for a subregion $\mathcal{P} \in \Pi$ can be expressed as the following line integral:

$$\delta(\mathcal{P}, t) = \int_{\mathcal{P}\cap\Sigma_t} f(x_1, x_2, t) V_n(x_1, x_2, t)\, \mathrm{d}s, \tag{2.13}$$

where the function f, defined on the twin boundary Σ_t, is called the driving traction and can be shown to take the form:

$$f(x_1, x_2, t) = -[\![w]\!] + \tfrac{1}{2}[\![u_{,\alpha}]\!](\sigma^+_{3\alpha} + \sigma^-_{3\alpha}), \tag{2.14}$$

where the superscripts indicate the limits as Σ_t is approached from two different sides. In this paper, we choose the unit normal **n** of the twin boundary to be pointing into the "$-$" side. Therefore, the normal velocity of the twin boundary V_n is positive if the twin boundary is moving toward the "$-$" side. For a twin boundary between variants $\mathcal{S}_0$ and $\mathcal{S}_1$, the driving traction specializes to

$$f = \frac{\mu\xi}{2}(u^+_{,2} + u^-_{,2} - \xi) = \frac{\xi}{2}(\sigma^+_{32} + \sigma^-_{32}). \tag{2.15}$$

In what follows, we will choose the $\mathcal{S}_0$ variant as the "$-$"side; unit normal **n** pointing into the "$-$" side. In a more general setting for thermoelastic materials [23], the energy dissipation rate in (2.13) is shown to be proportional to the entropy production rate, which is required to be non-negative by the second law of thermodynamics, $\delta(\mathcal{P}, t) \geq 0$. It follows from localization that the following inequality must hold:

$$f V_n \geq 0 \qquad \text{on } \Sigma_t. \tag{2.16}$$

This condition restricts the directions a twin boundary can move.

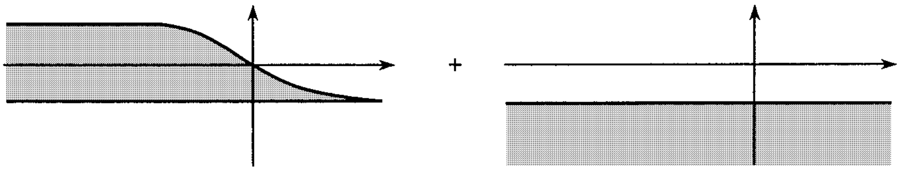

Figure 2. Decomposition of a twinning step into two parts: a semi-infinite twin needle and a perfect twin boundary with piecewise linear displacement.

3. Fundamental solutions

In the case of stationary twin boundaries, the equilibrium takes the form

$$\nabla^2 u = 0 \qquad \text{on} \Pi - \Sigma_t. \tag{3.1}$$

Consider twin boundaries formed by variant $\mathcal{S}_0$ and $\mathcal{S}_1$ such that the transformation strain is $\xi\mathbf{e}_2$. The jump conditions (2.12) on Σ_t become

$$\begin{cases} [\![u_{,\alpha}]\!] l_\alpha = 0, \\ [\![u_{,\alpha}]\!] n_\alpha = \xi n_2 \end{cases} \qquad \text{on } \Sigma_t. \tag{3.2}$$

A displacement field associated with a bounded twinned region $\mathcal{D}$ is found [24, 14] to be

$$u(\mathbf{x}) = \frac{-\xi}{2\pi} \int_{\partial\mathcal{D}} \log|\mathbf{x} - \mathbf{z}| n_2(\mathbf{z})\, \mathrm{d}s_{\mathbf{z}}. \tag{3.3}$$

The above displacement field approaches zero as $\mathbf{x}$ goes far away from the bounded region $\mathcal{D}$. The material inside $\mathcal{D}$ is of variant $\mathcal{S}_1$, while outside of $\mathcal{D}$ the material is in variant $\mathcal{S}_0$. The displacement in (3.3) satisfies (3.1) and the jump conditions (3.2). In order to have strains confined in the respective variants, the unit normal of the the twin boundary $\partial\mathcal{D}$ has to be close enough to the direction of the transformation strain $\mathbf{e}_2$. This restriction imposes severe conditions on the shape of $\mathcal{D}$ [24, 14]: Only slender, needle-like regions with the outward normal $\mathbf{n}$ close in direction to the $\mathbf{e}_2$ direction are possible. This is in good agreement with experimental observation of twin needles. Besides these qualitative restrictions on $\mathcal{D}$, the exact shape of $\mathcal{D}$ is otherwise arbitrary.

The above displacement field (3.3) is modified [15] to obtain the displacement fields associated with a semi-infinite twin needle and a twinning step. In the latter problem, the infinite domain Π is separated by the curve $x_2 = s(x_1)$ into two parts, $\mathcal{M}$ and $\mathcal{D}$; each contains material in variants $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively. The shape of the boundary is assume to satisfy following properties:

$$\begin{cases} s(x) = -h & \text{for } x \geq d_0, \\ s(x) = h & \text{for } x \leq -d_0, \\ s'(x) < 0 & \text{for } |x| < d_0, \\ s'(\pm d_0) = 0. \end{cases} \tag{3.4}$$

The shape of the twin boundary has a curved part between $x_1 = \pm d_o$. Outside this region, the twin boundary is flat. Twin boundaries of this kind are referred to as twinning steps. A typical shape of s is shown in Figure 1. The displacement is given by the superposition of two displacements. One associates with a semi-infinite twin $\tilde{\mathcal{D}}$ and the other is a piecewise linear field constructed by the shear strains $\mathbf{0}$ and $\xi\mathbf{e}_2$. See Figure 2 for an illustration of the decomposition. Let $u_{\mathcal{D}}$ denote the displacement associated with the stationary twinning step (3.4). Then it is given by the superposition of two fields,

$$u_{\mathcal{D}} = u_h + u_{\tilde{\mathcal{D}}}, \tag{3.5}$$

where

$$u_h = \begin{cases} 0, & \text{for } x_2 \geq -h, \\ \xi(x_2 + h) & \text{for } x_2 < -h, \end{cases} \tag{3.6}$$

is the piecewise linear field associated with the flat boundary and

$$u_{\tilde{\mathcal{D}}}(\mathbf{x}) = -\frac{\xi}{2\pi} \lim_{R\to\infty} \int_{\partial\tilde{\mathcal{D}}\cap B_R} \log|\mathbf{x} - \mathbf{z}| n_2(\mathbf{z})\, \mathrm{d}s_{\mathbf{z}} \tag{3.7}$$

is associated with the semi-infinite twin needle (Figure 2). Here B_R is a circular region of radius R centered at the origin. It is shown that [15] the displacement field (3.5) satisfies the equilibrium (3.1) and the jump conditions (3.2). Furthermore, the shear strain γ_2 of for the twin step (3.4) has the following limiting values:

$$u^{\pm}_{\mathcal{D},2}(\mathbf{x}) \;=\; \frac{\xi}{2\pi} \lim_{R\to\infty} ⨍_{-R}^{R} \frac{s(z) - s(x_1)}{(z - x_1)^2 + [s(z) - s(x_1)]^2}\, \mathrm{d}z + \frac{\xi}{2}[1 \pm n_2(\mathbf{x})] \qquad \text{on } \Sigma_t. \tag{3.8}$$

Note that the '+' side is chosen to be the region with higher strains ($\mathcal{S}_1$) and the '$-$' side, $\mathcal{S}_0$; $\mathbf{n}$ pointing into the '$-$' side. The symbol '$⨍$' denotes the Cauchy principal value of an integral.

4. Quasi-steady-state motion

In this section, we consider the quasi-steady-state motion of a twin boundary defined by $x_2 = s(x_1, t)$ as depicted in Figure 1. We focus on the twinning deformation formed by the variants $\mathcal{S}_0$ and $\mathcal{S}_1$. The displacement u satisfies the wave equation (2.11) with the jump conditions on the twin boundary:

$$\begin{cases} [\![u]\!] = 0, \\ [\![u_{,\alpha}]\!] n_\alpha + \frac{V_n}{c^2}[\![\dot{u}]\!] = \xi n_2 \end{cases} \qquad \text{on } \Sigma_t. \tag{4.1}$$

The phase boundary is moving along the $\mathbf{e}_1$ direction with velocity $\mathbf{v} = V(x_1, t)\mathbf{e}_1$. We assume that this velocity is close to the average velocity $\mathring{V}$ in the sense that the motion of the boundary, observed in a moving frame with velocity $\mathring{V}\mathbf{e}_1$, can be approximated by a quasi-static motion; *i.e.*, the inertial effect with respect to the moving frame can be ignored. Let $\mathring{x}_1$ and $\mathring{x}_2$ be the coordinates in the moving frame with velocity $\mathring{V}\mathbf{e}_1$. Utilize the following change of variables

$$\mathring{x}_1 = x_1 - \mathring{V}t, \qquad \mathring{x}_2 = x_2 \tag{4.2}$$

and define the displacement in the moving frame by

$$\mathring{u}(\mathring{x}_1, \mathring{x}_2, t) \equiv u(x_1, x_2, t) \qquad \text{with } x_1 = \mathring{x}_1 + Vt, \quad x_2 = \mathring{x}_2. \tag{4.3}$$

Also, denote the shape of the twin boundary by $\mathring{\Sigma}_t$ in the moving frame, then we have

$$\mathring{\Sigma}_t = \{\mathring{\mathbf{x}} = \mathring{x}_1\mathbf{e}_1 + \mathring{x}_2\mathbf{e}_2 \mid \mathring{x}_2 = \mathring{s}(x_1, t)\} \tag{4.4}$$

in which the function $\mathring{s}$ is given by

$$\mathring{s}(\mathring{x}_1, t) = s(\mathring{x}_1 + \mathring{V}t, t). \tag{4.5}$$

Following (3.4), the function $\mathring{s}$ has the following properties:

$$\begin{cases} \mathring{s}(x, t) = -h & \text{for } \mathring{x} \geq d(t) \\ \mathring{s}(x, t) = h & \text{for } \mathring{x} \leq -d(t) \\ \mathring{s}_x(x, t) < 0 & \text{for } |\mathring{x}| < d(t) \\ \mathring{s}_x(\pm d(t), t) = 0. \end{cases} \qquad \text{for all } t. \tag{4.6}$$

The interval $(-d(t), d(t))$ represents the curved region of the twin boundary. Outside of this interval, the twin boundary is flat and aligned with the x_1 direction.

Note that the normal velocity of the twin boundary is given by

$$V_n = \frac{s_t}{\sqrt{1 + s_x^2}} = \frac{\mathring{s}_t - \mathring{V}\mathring{s}_x}{\sqrt{1 + \mathring{s}_x^2}} = \mathring{V}_n + \mathring{V}n_1 \approx \mathring{V}n_1 \tag{4.7}$$

where s_x and $\mathring{s}_x$ denote the partial derivatives with respect to the spatial variables, while s_t and $\mathring{s}_t$ are the partial derivatives with respect to time. Therefore, the underlying assumption for quasi-steady-state motion is that $|\mathring{s}_t| \ll |\mathring{s}_x|$. Substituting (4.3) in (2.11), and ignoring the inertial terms in the moving frame, we arrive at the quasi-steady-state equation:

$$(1 - \frac{\mathring{V}^2}{c^2})\mathring{u}_{,11} + \mathring{u}_{,22} = 0 \qquad \text{on } \Pi - \mathring{\Sigma}_t. \tag{4.8}$$

The jump conditions (4.1) become

$$\begin{cases} [\![\mathring{u}]\!] = 0, \\ (1 - \frac{\mathring{V}^2}{c^2})[\![\mathring{u}_{,1}]\!]n_1 + [\![\mathring{u}_{,2}]\!]n_2 = \xi n_2 \end{cases} \qquad \text{on } \mathring{\Sigma}_t. \tag{4.9}$$

Note that since the inertial terms are ignored, the variable t serves merely as a parameter for the motion. Equations (4.8) and (4.9) can be further simplified by rescaling the $\mathring{x}_1$ axis using

$$\overset{*}{x}_1 = \lambda\mathring{x}_1, \qquad \overset{*}{x}_2 = \mathring{x}_2; \qquad \text{with } \lambda = \frac{1}{\sqrt{1 - \mathring{V}^2/c^2}}. \tag{4.10}$$

Let the displacement in the rescaled moving frame be denoted by $\overset{*}{u}$:

$$\overset{*}{u}(\overset{*}{x}_1, \overset{*}{x}_2, t) = \overset{\circ}{u}(\overset{*}{x}_1/\lambda, \overset{*}{x}_2, t) \tag{4.11}$$

and the twin boundary $\overset{*}{\Sigma}_t$ be given by

$$\overset{*}{x}_2 = \overset{*}{s}(\overset{*}{x}_1, t) \equiv \overset{\circ}{s}(\overset{*}{x}_1/\lambda, t). \tag{4.12}$$

It can be shown that the displacement $\overset{*}{u}$ satisfies

$$\overset{*}{u}_{,11} + \overset{*}{u}_{,22} = 0 \qquad \text{on } \Pi - \overset{*}{\Sigma}_t \tag{4.13}$$

with the jump conditions

$$\begin{cases} [\![\overset{*}{u}]\!] = 0, \\ [\![\overset{*}{u}_{,1}]\!]\overset{*}{n}_1 + [\![\overset{*}{u}_{,2}]\!]\overset{*}{n}_2 = \xi \overset{*}{n}_2 \end{cases} \qquad \text{on } \overset{*}{\Sigma}_t. \tag{4.14}$$

Here, $\overset{*}{n}_\alpha$ is the components of the unit normal to the boundary $\overset{*}{\Sigma}_t$. Equations (4.13) and (4.14) are identical in form to (3.2) and (3.3). Hence, in the rescaled moving frame, the quasi-steady state motion of a twin boundary takes the form of a one-parameter family of equilibrium field. The displacement is then given by

$$\overset{*}{u}(\overset{*}{x}_1, \overset{*}{x}_2, t) = u_{\overset{*}{\mathcal{D}}_t}(\overset{*}{x}_1, \overset{*}{x}_2) \tag{4.15}$$

where $u_{\overset{*}{\mathcal{D}}_t}$ is the fundamental field given by (3.5), (3.6) and (3.7), with the region $\mathcal{D}$ replaced by $\overset{*}{\mathcal{D}}_t$. According to (3.8), the shear strain γ_2 at the boundary is given, in the moving frame, by

$$\begin{aligned} \gamma_2 \;=\; & \frac{\lambda\xi}{2\pi} \lim_{R\to\infty} \fint_{-R}^{R} \frac{\overset{\circ}{s}(\overset{\circ}{z}, t) - \overset{\circ}{s}(\overset{\circ}{x}_1, t)}{\lambda^2(\overset{\circ}{z} - \overset{\circ}{x}_1)^2 + [\overset{\circ}{s}(\overset{\circ}{z}, t) - s(\overset{\circ}{x}_1, t)]^2} \, \mathrm{d}\overset{\circ}{z} \\ & + \frac{\xi}{2}(1 \pm \frac{1}{\sqrt{1 + \overset{\circ}{s}{}_x^2(x_1, t)/\lambda^2}}) \qquad \text{on } \overset{\circ}{\Sigma}_t. \end{aligned} \tag{4.16}$$

In view of (2.15), the driving traction at the boundary is given by

$$f = \frac{\lambda\mu\xi^2}{2\pi} \lim_{R\to\infty} \fint_{-R}^{R} \frac{\overset{\circ}{s}(\overset{\circ}{z}, t) - \overset{\circ}{s}(\overset{\circ}{x}_1, t)}{\lambda^2(\overset{\circ}{z} - \overset{\circ}{x}_1)^2 + [\overset{\circ}{s}(\overset{\circ}{z}, t) - \overset{\circ}{s}(\overset{\circ}{x}_1, t)]^2} \, \mathrm{d}\overset{\circ}{z} \qquad \text{on } \overset{\circ}{\Sigma}_t. \tag{4.17}$$

The stress field associated with (4.15) has vanishing remote stresses. If a constant remote shear stress $\sigma_0 \mathbf{e}_2$ is present, the shear strain field γ_2 is

$$\begin{aligned} \gamma_2 \;=\; & \frac{\lambda\xi}{2\pi} \lim_{R\to\infty} \fint_{-R}^{R} \frac{\overset{\circ}{s}(\overset{\circ}{z}, t) - \overset{\circ}{s}(\overset{\circ}{x}_1, t)}{\lambda^2(\overset{\circ}{z} - \overset{\circ}{x}_1)^2 + [\overset{\circ}{s}(\overset{\circ}{z}, t) - \overset{\circ}{s}(\overset{\circ}{x}_1, t)]^2} \, \mathrm{d}\overset{\circ}{z} \\ & + \frac{\xi}{2}(1 \pm \frac{1}{\sqrt{1 + \overset{\circ}{s}{}_x^2(x_1, t)/\lambda^2}}) + \gamma_0 \qquad \text{on } \overset{\circ}{\Sigma}_t. \end{aligned} \tag{4.18}$$

where $\gamma_0 = \sigma_0/\mu$ is the excess remote shear strain. Consequently, the driving force becomes

$$f = \frac{\lambda\mu\xi^2}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathring{s}(\mathring{z},t) - \mathring{s}(\mathring{x}_1,t)}{\lambda^2(\mathring{z}-\mathring{x}_1)^2 + [\mathring{s}(\mathring{z},t) - \mathring{s}(\mathring{x}_1,t)]^2}\, d\mathring{z} + \mu\xi\gamma_0 \quad \text{on } \mathring{\Sigma}_t. \tag{4.19}$$

5. Kinetic relation

In the previous section, the displacement field is obtained once the position of the twin boundary is specified. Apart from the slope restriction on s such that the strains on both sides will be confined to the respective variants, the motion of s is not determined. With the above constitutive model, the remote stress σ_0 fails to determine uniquely the motion of the twin boundary. A similar loss of uniqueness is encountered by Abeyaratne and Knowles [17] in a one-dimensional problem. To remedy this situation, a kinetic relation which relates the normal velocity of the twin boundary to the driving traction will be imposed. In particular, we are interested in the consequence of the following kinetic relation

$$V_n = Kf|n_1|. \tag{5.1}$$

This is first proposed by Rosakis and Tsai [16] in the study of steady state motion of twin boundaries. This relation models the motion of a twin boundary in which the normal velocity is proportional to the driving traction with fixed normal direction. The velocity increases when the normal direction deviates from the preferred direction $\mathbf{e}_2$. A detailed study by Tsai [15] showed that in the case of subsonic propagation ($\mathring{V} < c$), the steady state motion of a twin step is not possible for a wide class of kinetic relations including (5.1). In what follows, we will study the quasi-steady-state motion of a twinning step under the influence of applied (remote) stress $\sigma_0 = \mu\gamma_0$ with subsonic average propagation speed ($\mathring{V} < c$). Notice that

$$V_n = \mathring{V} n_1 + \frac{\mathring{s}_t}{\sqrt{1+\mathring{s}_x^2}} \qquad \text{with } \mathring{n}_1 = -\frac{\mathring{s}_x}{\sqrt{1+\mathring{s}_x^2}}. \tag{5.2}$$

Substitute (4.19) and (5.2) in (5.1) and choose $V = K\mu\xi\gamma_0$ to find

$$\mathring{s}_t = -\frac{\lambda\mu\xi^2 K}{2\pi} s_x \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathring{s}(\mathring{z},t) - \mathring{s}(\mathring{x}_1,t)}{\lambda^2(\mathring{z}-\mathring{x}_1)^2 + [\mathring{s}(\mathring{z},t) - \mathring{s}(\mathring{x}_1,t)]^2}\, d\mathring{z}. \tag{5.3}$$

Utilizing the fact that the slope of $\mathring{s}$ is required to be small, we can approximate the above equation by (see Appendix A)

$$\mathring{s}_t = -\frac{\mu\xi^2 K}{2\pi\lambda} \mathring{s}_x \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathring{s}_x(\mathring{z},t)}{(\mathring{z}-\mathring{x}_1)}\, d\mathring{z} = -\frac{\mu\xi^2 K}{2\pi\lambda} \mathring{s}_x \int_{-d(t)}^{d(t)} \frac{\mathring{s}_x(\mathring{z},t)}{(\mathring{z}-\mathring{x}_1)}\, d\mathring{z}$$

$$\text{for } \mathring{x}_1 \in (-d(t), d(t)). \tag{5.4}$$

Here we recognize the fact that $\mathring{s}_x = 0$ in the flat portion of the twin boundary. In addition, by virtue of the kinetic relation (5.1) (also reflected by (4.6)), the normal velocity of the twin boundary should be zero on the flat portion.

Equation (5.4) is a nonlinear integro-differential equation that governs the motion of the twin boundary characterized by the function $\mathring{s}$. The following theorem establishes a self-similar solution to this equation.

THEOREM 1. *A solution to (5.4) is given by*

$$\mathring{s}(\mathring{x}_1, t) = g(\frac{\mathring{x}_1}{d(t)}) \tag{5.5}$$

where the function g *is defined by*

$$g(\eta) = \begin{cases} +h & \text{for} \quad \eta < -1, \\ -\dfrac{2h}{\pi}\left[\arcsin(\eta) + \eta\sqrt{1-\eta^2}\right] & \text{for} \quad -1 \leq \eta \leq 1, \\ -h & \text{for} \quad \eta > 1; \end{cases} \tag{5.6}$$

and

$$d(t) = \sqrt{d_0^2 + 2Dt}, \qquad D = \frac{2K\mu\xi^2 h}{\lambda\pi}. \tag{5.7}$$

Proof. Let $|\mathring{x}_1| < d(t)$ be the region of the twin boundary with non-zero slope. Assume the function $\mathring{s}$ take the form of (5.5). Substitute into (5.4) and separate the variables to find that

$$d(t)\dot{d}(t) = \frac{\mu\xi^2 K}{2\pi\lambda}\frac{1}{\eta}\,⨍_{-1}^{1} \frac{g'(\zeta)}{(\zeta-\eta)}\,\mathrm{d}\zeta = D, \qquad \eta = \mathring{x}_1/d(t) \tag{5.8}$$

where D is an unspecified constant. It follows that $d(t)$ takes the form of $(5.7)_1$ with d_0 being the initial value of d at $t = 0$. In addition, g satisfies $g(\eta) = \mp h$ for $\pm\eta > 1$ and

$$⨍_{-1}^{1} \frac{g'(\zeta)}{(\zeta-\eta)}\,\mathrm{d}\zeta = \frac{2\pi\lambda D}{\mu\xi^2 K}\eta \qquad \text{for} \quad |\eta| < 1. \tag{5.9}$$

Equation $(5.6)_2$ follows from the identity (see Appendix B)

$$⨍_{-1}^{1} \frac{\sqrt{1-\zeta^2}}{(\zeta-\eta)}\,\mathrm{d}\zeta = -\pi\eta \tag{5.10}$$

and the boundary conditions $g'(\pm 1) = 0$ and $g(\pm 1) = \mp h$. This completes the proof.

The self-similar solution in (5.5) gives a monotonous shape of the twin boundary with negative slopes. The shape evolves in the fashion that the curved region grows while the slopes flatten. The unit normal of the twin boundary tends to the preferred direction $\mathbf{e}_2$. The length of the curved region is given by $2d(t) = 2\sqrt{d_0^2 + 2Dt}$ in (5.7). In the curved portion of twin boundary ($|\mathring{x}_1| < d(t)$), the difference between the actual propagating velocity along x_1 and the average velocity $\mathring{V}$ can be easily found to be

$$\begin{aligned} V &= \mathring{V} - \frac{\mathring{s}_t}{\mathring{s}_x} = \mathring{V} + \frac{\mathring{x}_1 D}{d^2} \\ &= \mathring{V} + \frac{\mathring{x}}{d}\frac{D}{\sqrt{d_0^2 + 2Dt}} \to \mathring{V} = K\xi\sigma_0 \qquad \text{as } t \to \infty. \end{aligned} \tag{5.11}$$

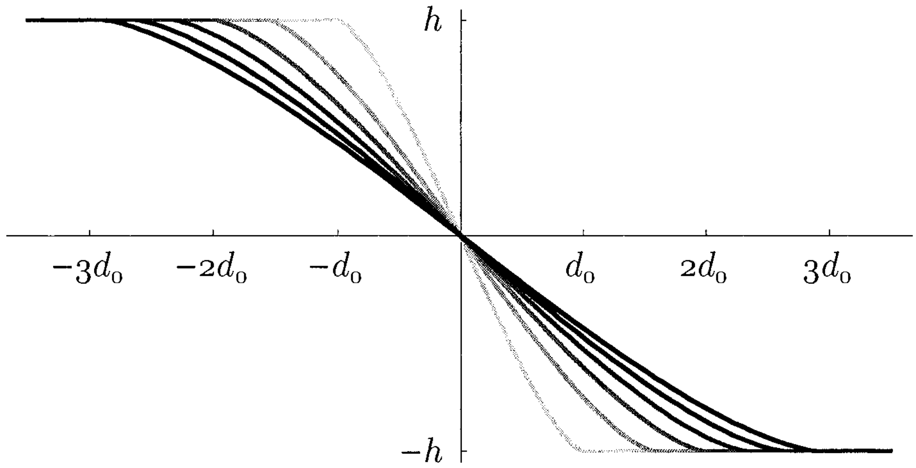

Figure 3. The evolution of the twinning step with initial shape given by the self-similar solution (5.6). The twinning boundaries are drawn darker as time increases. (The axes shown are $\mathring{x}_1$ and $\mathring{x}_2$.)

Therefore, the propagation velocity approaches the average value $\mathring{V}$ which is proportional to the applied shear stress σ_0. Figure 3 shows the shapes of a evolving twinning step at different times. Notice that the above solution (5.5) admits only the special shape given by $(5.6)_2$. Namely, if the initial shape of the twin boundary is not of the form given by g in (5.6), the evolution of the shape might differ dramatically from the above solution.

6. Linearization of the evolution equation

In this section, we perform the linearization of the evolution equation (5.4) for a twinning step with initial shape close to that of the self-similar solution (5.6). Consider the twin boundary characterized by the following

$$\mathring{s}(\mathring{x}_1, t) = g(\eta) + \varepsilon p(\eta, t) \tag{6.1}$$

with

$$\eta = \mathring{x}_1/d(t), \qquad d(t) = \sqrt{d_0^2 + 2Dt}, \qquad D = 2K\mu\xi^2 h/\lambda\pi. \tag{6.2}$$

Here p is the variation from the self-similar solution g given by (5.6), ε is a small parameter. Substitute (6.1) in the evolution equation (5.4) and take the first-order terms to find the linearized equation for p:

$$-D\eta p_\eta + d(t)^2 p_t(\eta, t) = -\frac{D}{4h} g'(\eta) \int_{-\infty}^{\infty} \frac{p_\eta(\zeta, t)}{\zeta - \eta} \,\mathrm{d}\zeta + \frac{D}{\pi} \int_{-1}^{1} \frac{\sqrt{1-\zeta^2}}{\zeta - \eta} \,\mathrm{d}\zeta. \tag{6.3}$$

Using the following identity (Appendix B)

$$\int_{-1}^{1} \frac{\sqrt{1-\zeta^2}}{\zeta - \eta} \,\mathrm{d}\zeta = \begin{cases} -\pi\eta & \text{for} \quad |\eta| < 1, \\ -\pi(\eta - \sqrt{\eta^2 - 1}) & \text{for} \quad \eta > 1, \\ -\pi(\eta + \sqrt{\eta^2 - 1}) & \text{for} \quad \eta < -1, \end{cases} \tag{6.4}$$

one finds the linearized evolution equation for p in three cases:

$$d(t)^2 p_t(\eta, t) = \frac{D}{\pi}\sqrt{1-\eta^2} \int_{-\infty}^{\infty} \frac{p_\eta(\zeta, t)}{\zeta - \eta}\, \mathrm{d}\zeta \qquad \text{for } |\eta| \le 1, \tag{6.5}$$

$$d(t)^2 p_t(\eta, t) = D\sqrt{\eta^2 - 1}\; p_\eta(\eta, t) \qquad \text{for } \eta > 1, \tag{6.6}$$

and

$$d(t)^2 p_t(\eta, t) = -D\sqrt{\eta^2 - 1}\; p_\eta(\eta, t) \qquad \text{for } \eta < -1. \tag{6.7}$$

The initial condition is given by

$$p(\eta, 0) = p_0(\eta) \equiv [\mathring{s}(d_0\eta, 0) - g(\eta)]/\varepsilon. \tag{6.8}$$

The equations for $|\eta| > 1$ can be solved by use of the following characteristics

$$\eta + \sqrt{\eta^2 - 1} = (\tau + \sqrt{\tau^2 - 1})\frac{d_0}{d(t)}, \qquad \tau > 1 \tag{6.9}$$

and

$$\eta - \sqrt{\eta^2 - 1} = (\tau - \sqrt{\tau^2 - 1})\frac{d_0}{d(t)}, \qquad \tau < -1. \tag{6.10}$$

Along each characteristic line (starting from the point $(\eta, t) = (\tau, 0)$) the value of the function p remains constant. This leads to the solution of p for $|\eta| > 1$:

$$p(\eta, t) = \begin{cases} p_0\left(\frac{d(t)}{2d_0}(\eta + \sqrt{\eta^2 - 1}) + \frac{d_0}{2d(t)}(\eta - \sqrt{\eta^2 - 1})\right) & \text{for } \eta > 1; \\ p_0\left(\frac{d(t)}{2d_0}(\eta - \sqrt{\eta^2 - 1}) + \frac{d_0}{2d(t)}(\eta + \sqrt{\eta^2 - 1})\right) & \text{for } \eta < -1. \end{cases} \tag{6.11}$$

Note that, in the (t,η) plane, the slopes of the characteristics are negative for $\eta > 1$ and positive for $\eta < -1$, respectively. (See Figure 4). It follows that if the initial values of p at $t = 0$ vanish outside the interval $[-1, 1]$, then $p(\eta, t) = 0$ for all $|\eta| > 1$ and $t > 0$. The support of the function $p(\eta, t)$ at any time will be always contained by $[-1, 1]$. Consequently, the curved part of the twinning step is always bounded by the interval $[-d(t), d(t)]$. For twinning steps with the curved part of the initial shape bounded, one can choose a suitable value of d_0 so that $p_0(\eta) = 0$ outside the interval $[-1, 1]$. We are left with the evolution equation for $|\eta| < 1$:

$$d(t)^2 p_t(\eta, t) = \frac{D}{\pi}\sqrt{1-\eta^2} \int_{-1}^{1} \frac{p_\eta(\zeta, t)}{\zeta - \eta}\, \mathrm{d}\zeta \qquad \text{for } |\eta| \le 1. \tag{6.12}$$

Notice the limits of the integral have been changed to account for the zero slope outside $[-1, 1]$. Separate variables by defining $p(\eta, t) = P(\eta)L(t)$. It follows from (6.12) that P and L are governed by

$$d^2(t)L'(t) = -c^2 L(t) \tag{6.13}$$

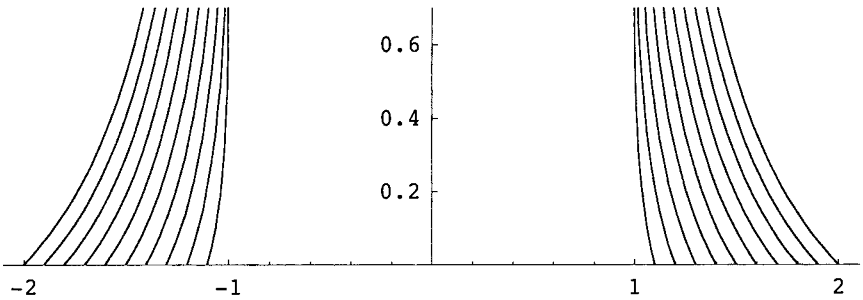

Figure 4. Characteristics of the linearized evolution equation for $|\eta| > 1$. (The axes shown are η and t.)

and

$$-\frac{c^2\pi P(\eta)}{D\sqrt{1-\eta^2}} = \frac{1}{\pi}\fint_{-1}^{1}\frac{P'(\zeta)}{\zeta-\eta}\,\mathrm{d}\zeta \qquad \text{for } |\eta| < 1, \tag{6.14}$$

where c is an nonzero constant. The solution to (6.13) with $L(0) = 1$ is given by

$$L(t) = \left(\frac{d_0}{d_0^2 + 2Dt}\right)^{c^2/2D} = \left[\frac{d_0}{d(t)}\right]^{c^2/D}. \tag{6.15}$$

Equation (6.14) can be transformed into an ordinary differential equation as follows. First, notice that the Cauchy principal value integral in (6.14) is the finite Hilbert transform of P'. Using the inverse transformation formula [25, pp. 13], one finds

$$\pi D P'(\eta) = \frac{c^2}{\sqrt{1-\eta^2}}\fint_{-1}^{1}\frac{P(\zeta)}{\zeta-\eta}\,\mathrm{d}\zeta. \tag{6.16}$$

Differentiate the above equation with respect to η and use (6.14) and (6.16) to find that P satisfies

$$(1-\eta^2)P' - \eta P' + \left(\frac{c^2}{D}\right)^2 P = 0. \tag{6.17}$$

This equation is satisfied by the Chebyshev polynomials [26, pp. 195]. However, in order to keep $P(\pm 1) = 0$, the solutions to (6.17) should be

$$P_k(\eta) = \begin{cases} \cos(k \arcsin(\eta)), & k = c^2/D = 1, 3, 5, \ldots \\ \sin(k \arcsin(\eta)), & k = c^2/D = 2, 4, 6, \ldots \end{cases} \tag{6.18}$$

Therefore, the following functions satisfy (6.12):

$$p_k(\eta, t) = P_k(\eta)L_k(t) \qquad \text{with } L_k(t) = \left[\frac{d_0}{d(t)}\right]^k \qquad k = 1, 2, 3, \ldots \tag{6.19}$$

with $p_k(\pm 1, t) = 0$ for all t. The general solution to (6.12) can be constructed by the infinite series

$$p(\eta, t) = \sum_{k=1}^{\infty} A_k P_k(\eta) L_k(t). \tag{6.20}$$

Since $L_k(0) = 1$ for all k, the coefficients A_k can be determined by the initial value $p_0(\eta) \equiv p(\eta, 0)$ in the following expansion,

$$p_0(\eta) = \sum_{k=1}^{\infty} A_k P_k(\eta). \tag{6.21}$$

Similar to the Chebyshev polynomials, $P_k(\eta)$ can be shown to be orthogonal on the interval $(-1, 1)$ with respect to the weight function $1/\sqrt{1-\eta^2}$. The coefficients A_k are then given by the following formula

$$A_k = \frac{\int_{-1}^{1} p_0(\eta) P_k(\eta)(\sqrt{1-\eta^2})^{-1}\, \mathrm{d}\eta}{\int_{-1}^{1} P_k^2(\eta)(\sqrt{1-\eta^2})^{-1}\, \mathrm{d}\eta} = \frac{2}{\pi} \int_{-1}^{1} p_0(\eta) P_k(\eta) \frac{1}{\sqrt{1-\eta^2}}\, \mathrm{d}\eta. \tag{6.22}$$

The above expansion is based on the restriction that P_k must vanish at the end points $\eta = \pm 1$. This requirement is implied by the condition that the growth of the curved portion of the twin boundary must be contained by the interval $(-d(t), d(t))$. However, the slopes of P_k in (6.18) approach infinity as η tending to ± 1. Since the linearized equation (6.5) is derived under the assumption of small slope of p, this result indicates that the higher-order terms play a significant role in the evolution of the twinning steps. The stability of the self-similar solution can not be properly analyzed without a fully nonlinear analysis on (5.3).

7. Concluding remarks

A quasi-steady-state evolution of a twin boundary under the setting of finite, anti-plane shear deformations has been presented. A continuum mechanics framework that models the sharp twin boundaries as discontinuities in strains has been adopted. The displacements are everywhere continuous. A special kinetic equation relating the driving traction and the propagating velocity of the twin boundary is assumed. The anisotropy in the kinetic relation accounts for the directional preference in the twinning mechanism. Displacement field associated with an infinite twin boundary with a kink has been found. This special self-similar solution consists of a special initial shape and approaches but never achieves a steady state as time increases. The average propagation speed has been found to be determined by the remote shear stress. Linearization of the evolution equation revealed that, for small slope variation, solutions with arbitrary initial shapes tend to stay close to the self-similar solution. However, the small slope assumption can not be maintained near the end points of the curved portion of the propagating twin boundary.

The continuum model and especially the anisotropic kinetic relation employed in this paper could be used to investigate other types of two-dimensional deformations, such as plane deformations. It is anticipated that such solutions will be qualitative similar to the self-similar one presented here for anti-plane shear.

It remains unsettled whether the solutions with general initial shape will converge uniformly to the self-similar solution. Further investigation into the stability of the self-similar solution is required to clarify this issue.

Appendix A. Approximation of $\mathring{s}_t$ in (5.4)

We need to show that

$$I = \lim_{R\to\infty} \fint_{-R}^{R} \frac{\mathring{s}(\mathring{z},t)-\mathring{s}(\mathring{x}_1,t)}{\lambda^2(\mathring{z}-\mathring{x}_1)^2+[\mathring{s}(\mathring{z},t)-\mathring{s}(\mathring{x}_1,t)]^2}\mathrm{d}\mathring{z} \approx \lim_{R\to\infty} \fint_{-R}^{R} \frac{\mathring{s}_x(\mathring{z},t)}{\lambda^2(\mathring{z}-\mathring{x}_1)}\mathrm{d}\mathring{z}. \tag{A1}$$

First observe the following identity (omitting t):

$$\begin{aligned}\frac{\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)}{\lambda^2(\mathring{z}-\mathring{x}_1)^2+[\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)]^2} &= \frac{\mathring{s}_x(\mathring{z})(\mathring{z}-\mathring{x})}{\lambda^2(\mathring{z}-\mathring{x}_1)^2+[\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)]^2} \\ &\quad -\frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}\mathring{z}}\arctan\left[\frac{\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)}{\lambda(\mathring{z}-\mathring{x})}\right].\end{aligned} \tag{A2}$$

Integrate both sides with respect to $\mathring{z}$ from $-R$ to R in the sense of Cauchy principal value to find

$$I = \fint_{-R}^{R} \frac{\mathring{s}_x(\mathring{z})(\mathring{z}-\mathring{x})}{\lambda^2(\mathring{z}-x_1)^2+[\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)]^2} - \arctan\left[\frac{\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)}{\lambda(\mathring{z}-\mathring{x})}\right]\Bigg|_{-R}^{R}. \tag{A3}$$

Note that the last term vanished when R approaches infinity. Equation (A1) follows with the approximation:

$$\frac{\mathring{s}_x(\mathring{z})(\mathring{z}-\mathring{x})}{\lambda^2(\mathring{z}-\mathring{x}_1)^2+[\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)]^2} = \frac{\mathring{s}_x(\mathring{z})}{(\mathring{z}-\mathring{x})}\frac{1}{\lambda^2+\left[\dfrac{\mathring{s}(\mathring{z})-\mathring{s}(\mathring{x}_1)}{(\mathring{z}-\mathring{x}_1)}\right]^2} \approx \frac{\mathring{s}_x(\mathring{z})}{\lambda^2(\mathring{z}-\mathring{x})}. \tag{A4}$$

Appendix B. Evaluation of the integral (5.10)

Let I be the integral in (5.10). Define $\zeta = \sin\theta$ and $\alpha = \pi - \theta$. It can be shown that

$$I = \fint_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{\sin\theta-\eta}\mathrm{d}\theta = \fint_{\pi/2}^{3\pi/2} \frac{\cos^2\alpha}{\sin\alpha-\eta}\mathrm{d}\alpha = \frac{1}{2}\fint_{0}^{2\pi} \frac{\cos^2\theta}{\sin\theta-\eta}\mathrm{d}\theta. \tag{B1}$$

The integral I can be expressed as a contour integral about the unit circle on the complex plane by use of the complex variable $z = \mathrm{e}^{\mathrm{i}\theta}$,

$$I = \frac{1}{4}\oint \frac{(1+z^2)^2}{z^2(z^2-2\mathrm{i}\eta z-1)}\,\mathrm{d}z. \tag{B2}$$

The identity (6.4) can be founded readily by residue theorem.

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